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# BASIC DEFINITIONS, PROPERTIES AND APPLICATIONS OF FRACTIONAL DERIVATIVES

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**Abstract.** This paper presents a rigorous and systematic investigation of fractional-order derivatives and their role in the formulation and analysis of fractional differential equations. The main definitions of fractional calculus, including the Riemann–Liouville, Caputo, and Grünwald–Letnikov approaches, are examined within a unified theoretical framework. Their mathematical properties, structural differences, and areas of applicability are comparatively analyzed. Particular attention is devoted to the Caputo derivative due to its compatibility with classical initial and boundary conditions, which makes it especially suitable for physical and engineering applications. The fundamental properties of fractional derivatives, such as linearity, non-local behavior, memory effects, and their relationship with classical integer-order differentiation, are discussed in detail. Analytical solution methods for fractional differential equations are developed using the Laplace transform and the Mittag–Leffler function, which generalizes the exponential function in fractional dynamics. Step-by-step solution procedures for complex initial–boundary value problems are provided to ensure mathematical consistency and clarity. The obtained results demonstrate that fractional-order models provide enhanced accuracy and flexibility in describing anomalous diffusion, hereditary phenomena, and memory-dependent processes. The study confirms the effectiveness of fractional calculus as a powerful mathematical tool for modeling complex dynamical systems in applied sciences and engineering.

**Key words:** Fraction orderly derivatives, Caputo approach, Riemann–Liouville expansion, Laplace changes.

## Introduction

Latest in years complicated memory and hereditary traits has been processes modeling issues practical and theoretical in research separately relevance profession is doing. Classic whole orderly differential equations many technician and natural processes in description effective although they are in real systems observable far term memory effect, anomalous diffusion and late answer events complete express This view from the perspective of fraction orderly derivatives and they based on differential equations classic mathematician models generalizing strong tool as manifestation is happening. Fraction analysis in particular time and in space spread abnormal speed processes in description important importance Foreexample, daily in life occurring of heat uneven spread, soil and pore in environments of liquid slowly or accelerated filtration, polymer and viscoelastic of materials to deformation late answer such as events fraction orderly models through more precisely is expressed in biology. of drugs in the body absorption and spread, economic in processes and market system past to the circumstances related development fraction derivatives using effective is being modeled.

## Fraction orderly derivatives main definitions, properties.

### Lemma 1. Gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad R(z) > 0.$$

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Here  $z$  is a complex number  $R(z)$  whose real part.

**The gamma function recurrent property.**

$$\Gamma(z+1) = z \cdot \Gamma(z).$$

If  $z \in N$ :

$$\Gamma(z+1) = z!$$

$z$  is not a natural number situations for examples:

**Example 1:** Find what  $\Gamma\left(\frac{1}{2}\right)$ ?

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} dt = \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt$$

This integral solution for variable by replacing we get:

$$t = x^2, \quad 2xdx = dt.$$

From this below expression come it turns out:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x^2}} 2xdx = 2 \int_0^{\infty} \frac{e^{-x^2}}{x} xdx = 2 \int_0^{\infty} e^{-x^2} dx$$

Above integral Gaussian integral with tied we get:

**Lemma 2:** Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{ pair from functionality comes out.}$$

So

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-x^2} dx = 2 \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

**Example 2:** Find what  $\Gamma\left(\frac{3}{2}\right)$ ?

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

**Lemma 3. Beta function.**

If  $p > 0$  and  $q > 0$  then the Beta function through the following integral found:

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p > 0, q > 0.$$

Gamma and Beta functions connection (Euler theorem).

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

**Example 3.** Find what  $B\left(\frac{1}{2}, \frac{1}{2}\right)$ ?

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} = \frac{\sqrt{\pi}\sqrt{\pi}}{1} = \pi$$

**Example 4.** Find what  $B(1, 2)$ ?

$$B(1, 2) = \int_0^1 (1-t) dt = t - \frac{t^2}{2} \Big|_0^1 = \frac{1}{2}$$

**Riemann–Liouville fraction orderly product**

Definition. If  $n-1 < \alpha < n$  if, then:

$$D_{a+}^{\alpha} f(t) = \frac{d^n}{dt^n} (I^{n-\alpha} f(t))$$

namely:

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

**Caputo fraction product**

Definition. If  $n-1 < \alpha < n$  If, then the Caputo fraction orderly product as follows will be:

$${}^c D_{a+}^{\alpha} f(t) = I_{a+}^{n-\alpha} \left( \frac{d^n f}{dt^n} \right)$$

namely:

$${}^c D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

**Example 5.** Let, be a Riemann–Liouville fraction. orderly product through solve

$$f(t) = t^3 + 2t^2 + t - 1, \alpha \in (0,1)?$$

$$I^{n-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\tau^3 + 2\tau^2 + \tau - 1}{(t-\tau)^\alpha} d\tau$$

Integral linearity to the property separation:

$$I^{n-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left[ \int_0^t \frac{\tau^3}{(t-\tau)^\alpha} d\tau + 2 \int_0^t \frac{\tau^2}{(t-\tau)^\alpha} d\tau + \int_0^t \frac{\tau}{(t-\tau)^\alpha} d\tau - \int_0^t \frac{1}{(t-\tau)^\alpha} d\tau \right]$$

**Lemma 4.** The beta function of the integral through calculation

$$\int_0^t \frac{\tau^m}{(t-\tau)^\alpha} d\tau = t^{m+1-\alpha} \frac{\Gamma(m+1)\Gamma(1-\alpha)}{\Gamma(m+2-\alpha)}$$

Above from the lemma used without integrals is considered  $f(t)$  function degrees i.e. for  $m=3, m=2, m=1$  and  $m=0$  every one integral we count and following to the result we achieve:

$$I^{1-\alpha} f(t) = \Gamma(1-\alpha) \left[ \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \right]$$

Each limit we will produce and Riemann–Liouville fraction orderly to the product has we will be:

$$\begin{aligned} {}^{RL}D_{a+}^\alpha f(t) &= \frac{d^n}{dt^n} \left( \Gamma(1-\alpha) \left[ \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \right] \right) \\ &= \frac{6(4-\alpha)\Gamma(1-\alpha)}{\Gamma(5-\alpha)} t^{3-\alpha} + \frac{4(3-\alpha)\Gamma(1-\alpha)}{\Gamma(4-\alpha)} t^{2-\alpha} + \\ &+ \frac{(2-\alpha)\Gamma(1-\alpha)}{\Gamma(3-\alpha)} t^{1-\alpha} - \frac{(1-\alpha)\Gamma(1-\alpha)}{\Gamma(2-\alpha)} t^{-\alpha}. \end{aligned}$$

**Example 5.** Let, Caputo fraction orderly product through solve

$$f(t) = t^3 + 2t^2 + t - 1, \alpha \in (0,1)?$$

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau$$

Function derivative we will find and instead of let 's put

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{3\tau^2 + 4\tau + 1}{(t-\tau)^\alpha} d\tau$$

Linearity from the property we use

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left[ 3 \int_0^t \frac{\tau^2}{(t-\tau)^\alpha} d\tau + 4 \int_0^t \frac{\tau}{(t-\tau)^\alpha} d\tau + \int_0^t \frac{1}{(t-\tau)^\alpha} d\tau \right]$$

Using **Lemma 4.** integrals we count and following the result we get,

$${}^cD_t^\alpha f(t) = \frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{4}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}$$

**Conclusion**

Conclusion as in other words, Caputo and Riemann–Liouville fractions orderly derivatives above analytical in a way calculated. In the Caputo derivative permanent terms disappears, this elementary to the conditions suitable the solution provides, in the Riemann–Liouville derivative and The singular term is preserved. Both derivative Beta and Gamma (function) using simplified, fraction order integral and derivatives mathematician stagnation is provided.

**List of references**

[1]. Podlubny I., Fractional Differential Equations, Academic Press, San Diego, 1999, 340 p.  
 [2]. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., Theory and Applications of Fractional Differential Equations, Elsevier - Amsterdam, 2006, 523 p.  
 [3]. Samko, S.G., Kilbas, A.A., Marichev, O.I., Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, Amsterdam, 1993, 976 p.  
 [4]. Diethelm, K., The Analysis of Fractional Differential Equations, Springer, Berlin, 2010, 247 p.  
 [5]. Miller, K.S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993, 384 p.  
 [6]. Oldham, K.B. & Spanier, J. (1974). The fractional calculus: theory and applications of differentiation and integration to arbitrary order.

Mathematics in Science and Engineering, Vol. 111. Academic Press, New York.

[7]. Samko, S.G., Kilbas, A.A. & Marichev, O.I. (1993). Fractional integrals and derivatives: theory and applications. Gordon and Breach, New York.

[8]. Podlubny, I. (1999). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, Vol. 198. Academic Press, San Diego.

[9]. Kilbas, A.A., Srivastava, H.M. & Trujillo, J.J. (2006). Theory and applications of fractional differential equations. North-Holland Mathematics Studies, Vol. 204. Elsevier, Amsterdam.

[10]. Diethelm, K. (2010). The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type. Lecture Notes in Mathematics, Vol. 2004. Springer, Berlin.

[11]. Mainardi, F. (2010). Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models. Imperial College Press, London.

[12]. Atanacković, T.M., Pilipović, S., Stanković, B. & Zorica, D. (2014). Fractional calculus with applications in mechanics: vibrations and diffusion processes. ISTE-Wiley, London.

[13]. Li, C. & Cai, M. (2020). Theory and numerical approximations of fractional integrals and derivatives. Other Titles in Applied Mathematics, Vol. 163. SIAM, Philadelphia.

[14]. Agarwal, P., Cattani, C. & Momani, S. (eds.) (2024). Fractional differential equations: theoretical aspects and applications. Elsevier/Academic Press, Amsterdam.

[15]. Teso, F. & Gómez-Castro, D. (2025). "Fractional derivatives: an extension of classical analysis to non-integer orders". Gaceta de la Real Sociedad Española, translated from Spanish.